Spectra, Categories, Composition Law and Segal Condition

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This paper gives a method to associate a spectra to a category with some additional structure. Since each spectra corresponds to a generalized cohomology theory, in this way we can build a correspondence between these categories and generalized cohomology theories.

The approach used in this paper is generally based on the paper [4] by G. Segal, which uses the construction of Γ-space as the bridge in between.

Aside from the original intention of Segal to apply this approach to K-theory, another interesting point is that the requirements put on a special category called Γ to be introduced later, that makes it a "storehouse" of what rules a binary operation should obey, actually inspires many other versions of conditions which are put onto simplicial objects and some more abstract constructions alike, summarized by later literature as *Segal condition*. In the end of this paper, we present a presence of Segal condition under the modern topic of higher categories.

For simplicity, we use the term *space* to indicate the smaller category of compactly generated topological spaces as stated in [2].

1 Preliminaries

1.1 Spectra and generalized cohomology theories

We have already seen that there is a natural link between "ordinary" cohomology groups and homotopy groups, i.e. there is a natural isomorphism

$$H^n(-;G) \cong \langle -, K(G,n)\rangle \tag{1}$$

between functors running through the homotopy category of CW complexes.

Although the Eilenberg–Steenrod axiom pins down the behavior of any "qualified" cohomology theory, if we remove the dimension axiom, there will be a big family of different theories, including some interesting ones like reduced cohomology. This class of cohomology theories are called *generalized cohomology theory*. It turns out that each generalized cohomology theory has its own version of the natural isomorphism as in Eq 1. In this case, $K(G,n)$ is replaced by some other sequence of spaces called *spectrum*.

Definition 1. A (Ω-)spectrum is the following data:

1. a sequence of pointed spaces $\{E^n\}$ for $n \geq 0$;
2. base point preserving closed embedding $E^n \to \Omega E^{n+1}$ for $n \geq 0$.

These maps are called *structural maps* of the spectrum. We define the *loop spectrum* of a given spectrum $\mathbf{E}$ to be a sequence of colimits

$$(\omega\mathbf{E})_k := \lim_{\to} \Omega^n E^{n+k}$$

(There are many versions of spectrum. In this paper, we use *spectrum* to indicate the more classical version of Ω-spectrum.) The spectrum $\{K(G,n)\}$ mentioned before is a special case of (Ω)-spectrum called Eilenberg–Steenrod spectrum. This can be made precise by recalling that $K(G,n) = \Omega K(G,n+1)$. which are its structural maps.

To complete the picture, we introduce the suitable category to talk about spectra. The objects of this category are spectra, of course, and morphism between spectra defined as follows:

Definition 2. A morphism from spectrum $\mathbf{X}$ to $\mathbf{Y}$ is a sequence of maps $f_k : X_k \to (\omega Y)_k$ that makes the following diagram commute:

$$\begin{array}{ccc} X_k & \xrightarrow{f_k} & (\omega Y)_k \\ \downarrow & & \downarrow \\ \Omega X_{k+1} & \xrightarrow{\Omega f_{k+1}} & \Omega(\omega Y)_{k+1} \end{array}$$

This theorem makes solid that every generalized cohomology theory can be represented by a spectrum.

Theorem 1 (Brown)**.** *A functor F from the pointed homotopy category of spaces to the category of pointes sets is representable if and only if*

1. *it takes coproducts to products;*
2. *it takes weak pushouts to weak pullbacks.*

In the case of CW complexes, the second required property in the theorem is equivalent to: Pick a pair of CW pairs $((X, A_1), (X, A_2))$ such that $A_1 \cup A_2 = X$ and $A_1 \cap A_2$ is still a CW complex. Then for all $x_1 \in F(A_1)$, $x_2 \in F(A_2)$ such that both of them restricts to the same element of $F(Y \cap Z)$, there exists some $y \in F(X)$ such that y restricts to x_1 and x_2 on A_1 and A_2 respectively.

As a result, given a cohomology theory

$$h^* : \mathrm{CW}^{\mathrm{op}} \to \mathrm{Ab}, \tag{2}$$

we have spaces E^k such that

$$h^k(X) \cong [X, E^k]. \tag{3}$$

There is a special kind of spectrum that plays a important role in the following sections: sphere spectrum.

Definition 3. The *sphere spectrum* $\mathbf{S}$ is the suspension spectrum $\{S^0, \Sigma S^0 = S^1, \Sigma^2 S^0 = S^2, \dots\}$. This spectrum can also be made into an Ω-spectrum since Ω and Σ are adjoint to each other.

1.2 Classifying space of categories

When I first learned about category theory I wondered if I can treat a category as a "graph" or something, and use the usual tool in graph theory or topology to study it. I gave up immediately since I think a category is often too complicated in most cases to be studied in this way. However, this actually turns out to be natural and useful in this field of study.

We associate a CW-complex to a category C by putting a 0-cell for each object in C, a 1-cell for each morphism, a 2-cell for each commutative diagram in the shape of

$$\begin{array}{ccc} X_0 & \longrightarrow & X_2 \\ \downarrow & \nearrow & \\ X_1 & & \end{array}$$

and so on.

To formalize this idea, we first recall something about simplicial set. For convenience we define $\mathbf{n} := \{1, 2, \ldots, n\}$ and $[n]$ to be $\mathbf{n} \cup \{0\}$ with additional natural order structure $<$.

Definition 4. A simplicial set is a contravariant functor $A : \mathsf{Ord}^{\mathrm{op}} \to \mathsf{Sets}$, where Ord denotes the category of finite totally ordered sets.

Such kind of construction can be understood intuitively by seeing the data of A as a sequence of sets $A_0, A_1, A_2, \ldots$ (where $A_i := [i]$), which can be seen as collection of geometric "parts" of different dimensions, and appropriate boundary maps and degeneracy maps that encodes how these parts should be glued together. When these boundary maps and degeneracy maps are consistent, we can use these recorded data as instructions to recover a space:

Definition 5. The *geometric realization* $|A|$ of simplicial space $A = \{A_n\}$ is defined to be

$$\left(\coprod_{n \geq 0} \Delta^n \times A_n \right) / \sim$$

where Δ^n is the standard n simplex and

Definition 6. Let C be a category. We define the *nerve* $N\mathsf{C}$ of C to be a semi-simplicial set defined by

$$N\mathsf{C}(S) := \mathrm{Functors}(S; \mathsf{C}).$$

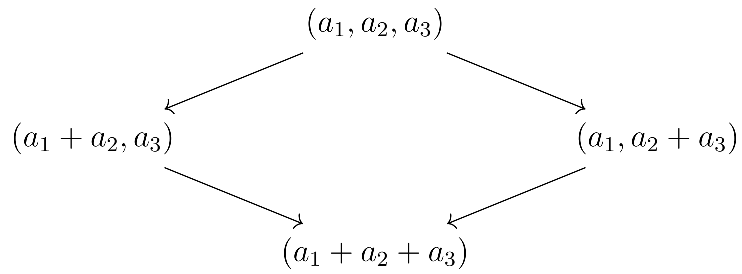
where $\mathrm{Functors}(S; \mathsf{C})$ denotes the set of functors from the "order category" of ordered set S to category C. Next we define the *classifying space* of C to be the geometric realization of semi-simplicial set $N\mathsf{C}$ and denote it by $B\mathsf{C}$.

2 Γ-spaces

The motivation of a Γ-space is to somehow relax the associativity condition of a topological abelian group. (Working directly with topological abelian group is somehow too "restricted" since every topological abelian group is weakly homotopy equivalent to a product of Eilenberg-MacLane spaces, which can be seen easily from the structural theorem of abelian groups and the fact that homotopy group functors respects products.) As a example, we can describe the usual associativity law of abelian groups in an "overkill" approach: for abelian group A, let θ be a map that assigns a subset of $\{1, 2, \ldots, m\}$ to a given number $i \in \{1, 2, \ldots, n\}$. Define $\theta^* : A^n \to A^m$ by

$$\theta^*(a_1, \ldots, a_n) := (b_1, \ldots, b_m), \quad \text{where } b_i := \sum_{j \in \theta(i)} a_j$$

These θ^* forms a graph with vertices being A^n for different n (the case for a abelian group with 3 elements is visualized below). Then the associativity is described by requiring this graph to be commutative.



Motivated by this example, we have the following definitions:

Definition 7. Define Γ to be the category that

1. its objects are all finite sets;
2. morphisms from finite set S to T are the maps $\theta : S \to \mathscr{P}(T)$, (where $\mathscr{P}(T)$ denotes the power set of T) such that $\theta(\alpha)$ and $\theta(\beta)$ are disjoint for distinct α and β.
3. Define the composition of $\theta : S \to \mathscr{P}(T)$ and $\phi : T \to \mathscr{P}(U)$ to be $\psi : S \to \mathscr{P}(U)$, where $\psi(\alpha) := \cup_{\beta \in \theta(\alpha)} \phi(\beta)$.

This Γ is the object that encodes all information about the composition law. Then if we attach this Γ to a space, we have an actual space with some operation that obeys the composition law recorded in Γ:

Definition 8. Define a Γ-space to be a contravariant functor $A : \Gamma^{\text{op}} \to \mathsf{Top}$ such that

1. $A(\mathbf{0}) = A(\varnothing)$ is contractible;
2. for any integer n, the map $p_n : A(\mathbf{n}) \to A(\mathbf{1})^n$ defined by
$$p_n = (A(i_1), A(i_2), \ldots, A(i_n))$$
(where $i_k : \mathbf{1} \to \mathbf{n} \in \operatorname{Mor}(\Gamma)$ is defined by $i(1) := \{k\}$) is a homotopy equivalence.

We call $A(1)$ to be the *underlying space* of Γ-space A. The good thing about this Γ is that it acts like a "detachable module" and it turns out that we can attach it to something else, for example a category, to apply the same composition law. This gives us the parallel concept of a "Γ-category", which turns out to be connected to the concept of Γ-space by taking classifying space of categories, which we shall discuss in the next section.

Next we demonstrate how can we obtain a spectrum from a given Γ-space. We can see that the Γ-space is a semi-simplicial space, and we can consider its geometric realization. This enables the definition:

Definition 9. For Γ-space A, define its *classifying space* BA to be a Γ-space such that $BA(S)$ is the realization of the Γ-space $T \mapsto A(S \times T)$.

Since geometric realization respects homotopy equivalence and products, we can see that BA satisfies the definition of a Γ-space, which makes this definition valid.

If we recursively apply the operation of finding the classifying space, we will get a spectrum $A(\mathbf{1}), BA(\mathbf{1}), B^2A(\mathbf{1}), \ldots$. We denote it by $\mathbf{B}A$.

In the rest of this section we discuss some nice properties of such construction. We can use the information brought by Γ to turn $A(\mathbf{1})$ into an H-space in the following way: consider the composition

$$A(\mathbf{1}) \times A(\mathbf{1}) \xrightarrow{p_2^{-1}} A(\mathbf{2}) \xrightarrow{A(m_2)} A(\mathbf{1}) \tag{4}$$

where p_2^{-1} is the homotopy inverse of p_2 defined in the definition of Γ and $m_2 : \mathbf{1} \to \mathbf{2}$ is a morphism in Γ sending 1 to $\{1, 2\}$.

Proposition 1. Suppose A is a Γ-space and its underlying space $A(\mathbf{1})$ is k-connected. Then $BA(\mathbf{1})$ is $(k+1)$-connected. And $A(\mathbf{1}) = \Omega BA(\mathbf{1})$ if and only if the H-space structure on $A(\mathbf{1})$ has homotopy inverse.

This proposition can be restated more generally by extracting the underlying simplicial spaces:

Proposition 2. Suppose $[n] \mapsto A_n$ be a simplicial space such that

1. A_0 is contractible;
2. $p_n : A_n \to A_1^n := (i_1^*, i_2^*, \dots, i_n^*)$ is a homotopy equivalence, where $i_k : [1] \to [n]$ sends 0 to $k-1$ and 1 to k,

then

1. if A_1 is k-connected, then $|A|$ is $(k+1)$-connected;
2. The adjunction pair $A_1 \to \Omega|A|$ and $SA_1 \to |A|$ is a homotopy equivalence iff A_1 has a homotopy inverse.

Now we discuss the converse direction, i.e. how to get a Γ-space from a spectrum. It turns out that this operation is adjoint with the operation $\mathbf{B}(-)$ of finding the classifying space.

Definition 10. Let $\mathbf{X}$ be a spectrum. Define the Γ-space $\mathbf{AX}$ corresponding to $\mathbf{X}$ to be $\mathbf{n} \mapsto (\mathbf{AX})(\mathbf{n}) := \mathrm{Mor}(\mathbf{S}^n; \mathbf{X})$, where $\mathbf{S}$ is the sphere spectrum defined previously.

To qualify $\mathbf{AX}$ as a Γ-space, we only need to check that $p_n : \mathbf{AX}(\mathbf{n}) \to \mathbf{AX}(\mathbf{1})^n$. This can be done by noticing the equivalence

$$(\mathbf{AX})(\mathbf{n}) = \mathrm{Mor}(\mathbf{S}^n; \mathbf{X}) \simeq \mathrm{Mor}(\vee_{i=1}^n \mathbf{S}; \mathbf{X}) = (\mathrm{Mor}(\mathbf{S}; \mathbf{X}))^n = ((\mathbf{AX})(\mathbf{1}))^n$$

3 Γ-Categories

The final piece of puzzle of the grand "Spectra–Γ-space–Category" correspondence is so called Γ-category. It is the natural counterpart of Γ-space introduced before.

Definition 11. Define a Γ-*Category* to be a contravariant functor $\mathscr{C}$ from Γ to Cats such that

1. $\mathscr{C}(\mathbf{0})$ is equivalent to the "singleton category" with one object and one morphism;
2. for each $n \geq 0$, the functor $p_n : \mathscr{C}(\mathbf{n}) \to (\mathscr{C}(\mathbf{1}))^n$ defined by $p_n = (\mathscr{C}(i_1), \dots, \mathscr{C}(i_n))$ (i_k is defined before in the definition of Γ-spaces) is equivalence of categories.

Immediately after this definition, we can see that if $\mathscr{C}$ is a Γ-category, the functor $|\mathscr{C}|$ defined by $S \mapsto |\mathscr{C}(S)|$ is a Γ-space.

We can construct a Γ-category from a category C where sum always exists. Suppose S is a finite set and $\mathscr{P}(S)$ is the power set of S made into a category by regarding morphisms to be the inclusion betwenn subsets. Define $\mathsf{C}(S)$ to be a category where objects are functors from $\mathscr{P}(S)$ to C that sends disjoint union to the sum in C, and morphisms are natural isomorphisms between functors. We can see that the $\mathsf{C}(-)$ we just construct is a contravariant functor from Γ to the category of categories. It is a Γ-category since $\mathsf{C}(\mathbf{2}) \to \mathsf{C} \times \mathsf{C}$ is a equivalence of categories, and we can extend this to the case for n by inductively unfold the statement.

Example. Start with the category Fin with disjoint union. Then $|\mathsf{Fin}(\mathbf{1})| = \coprod_{n\geq 0} BS_n$ where S_n is symmetric group. If we choose the underlying category to be the category of finite-dimensional real vector spaces and use the sum $\oplus$, then $A(\mathbf{1} = \coprod_{n\geq 0} B(\mathrm{GL}(n,\mathbb{R}))$. As we can see, the changing part is always the "symmetric group" of the corresponding category.

Now we have completed our landscape:

$$\begin{array}{c} \{\text{Spectra}\} \\ \mathbf{B}(-)\uparrow\downarrow\mathbf{A}(-) \\ \{\Gamma\text{-spaces}\} \\ \uparrow |-| \\ \{\Gamma\text{-categories}\} \end{array}$$

4 Segal Condition in Higher Categories

Apart from the case for Γ-space and Γ-category, the idea of using certain "object" under the restrictions we stated repeatedly in definition 8 and 11 to record the "compositing behavior" of a binary operation turns out to be pretty universal. In later literature, this idea is summarized as so called *Segal condition*. We present a appearance of this idea in the modern setting of higher categories, controlling the composition law of paring operation for a monoidal category.

Definition 12. A *monoidal category* M is a category equipped with a monoidal paring $\otimes : \mathsf{M}\times\mathsf{M} \to \mathsf{M}$ and a monoidal unit $\mathbb{S}\in\mathsf{M}$, satisfying proper associativity, left and right unitality in terms of natural isomorphisms.

Let M be a monoidal category. A new category $\mathsf{M}^{\otimes}$ can be constructed as:

1. objects are finite sequence $(M_1, M_2, \ldots, M_n)$ of objects in M;
2. morhpisms from $(M_1,\ldots,M_n)$ to $(L_1,\ldots,L_k)$ is a pair $(\alpha, \{f_i\}_i)$ where α is a map from $[k]$ to $[n]$, and $f_i : M_{\alpha(i-1)+1}\otimes\cdots\otimes M_{\alpha(i)} \to L_i$ for $i = 1,\ldots,k$.

Let $p : \mathsf{C}\to\mathsf{D}$ be a functor and $d\in\mathsf{D}$. Then we can define the the fiber C_d of p over d by the pullback diagram

$$\begin{array}{ccc} \mathsf{C}_d & \longrightarrow & \mathsf{C} \\ \downarrow & & \downarrow p \\ [0] & \xrightarrow{\mathrm{Const}_d} & \mathsf{D} \end{array}$$

It turns out that if p satisfy some technical condition (being a Grothendieck opfibration), then we can lift a morphism $\alpha : d_1 \to d_2 \in \mathrm{Mor}(\mathsf{D})$ to a morphism $\alpha_! : \mathsf{C}_{d_1}\to\mathsf{C}_{d_2}$ between fibers.

The projection functor $p : \mathsf{M}^{\otimes}\to\Delta^{\mathrm{op}}$ defined by sending $(M_1,\ldots,M_n)$ to $[n]$ and sending $(\alpha,\{f_i\}_i)$ to α is one of such morphism. Define *Segal maps* to be

$$\sigma : \mathsf{M}^{\otimes}_{[n]} \to \mathsf{M}^n := ((\iota_1)_!,\ldots,(\iota_n)_!) \tag{5}$$

where $\iota_i : [n]\to[1]\in\Delta^{op}$ is the opposite morphism of $0\mapsto i-1, 1\mapsto i$.

It turns out that these Segal maps are equivalences, and this property is called *Segal condition*. We can see that it is exactly the same thing as we used in definition 8 and 11.

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